

# The arithmetic of solids

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## Abstract

The set of segments, each of the next is  $n$  times bigger than the first one is a simple geometric interpretation of the set  $\mathbb{N}$  of natural numbers. In this paper we investigate the opposite situation. We construct an algebraic structure similar to the set  $\mathbb{N}$  which describes the set of congruent triangles, each of the next has the sides  $n$  times bigger than the first one. Later we do the same with the set of congruent tetrahedrons, and finally with a set of simplices of any dimension.

## Introduction

If we take the mapping from the set of natural numbers  $\mathbb{N}$  to the set of unit segment and  $n$  times longer segments then we will have some geometric interpretation of the set  $\mathbb{N}$ . If we replace segments by vectors we will get a geometric interpretation of integer numbers.

If now we take the fixed triangle and the set of congruent triangles which the sides are  $n$  times bigger than the fixed one's sides, we will have the question. Whether the set of the above triangles is the geometric interpretation of any algebraic structure other than set  $\mathbb{N}$ ? We try to answer this question in the first section. In the second one we investigate the set of congruent tetrahedrons. In the third section we do the same with the set of simplices of any dimension but in a modest range.

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# 1 The arithmetic of triangles

## 1.1 Basic concepts

Let us set the triangle on the plan  $\mathbb{R}^2$  and let us denote it by the symbol  $\langle 1 \rangle$ . Then the congruent triangle with  $n$  times bigger sides (we denote it by the symbol  $\langle n \rangle$ ) is built from  $\frac{n(n+1)}{2}$  triangles  $\langle 1 \rangle$  and  $\frac{n(n-1)}{2}$  symmetrical to triangle  $\langle 1 \rangle$  in relation to any side, triangle denoted by symbol  $\langle -1 \rangle$  (see Fig. 1). So

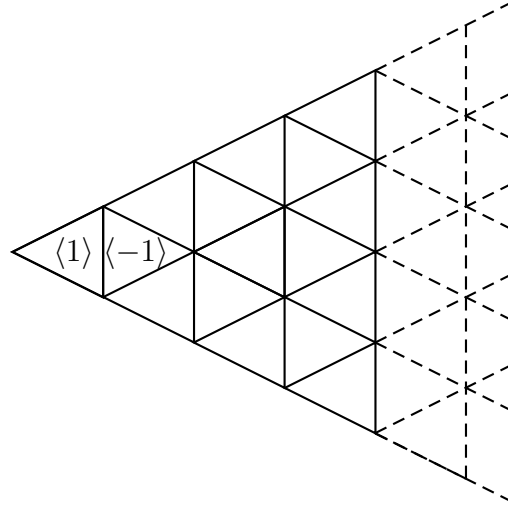


Fig. 1.

$$\langle n \rangle = \frac{n(n+1)}{2} \langle 1 \rangle + \frac{n(n-1)}{2} \langle -1 \rangle \quad (1)$$

Similarly let us mark by the symbol  $\langle -n \rangle$  the triangle congruent to the triangle  $\langle -1 \rangle$  with  $n$  times bigger sides. Then

$$\begin{aligned} \langle -n \rangle &= \frac{n(n-1)}{2} \langle 1 \rangle + \frac{n(n+1)}{2} \langle -1 \rangle \\ &= \frac{(-n)(-n+1)}{2} \langle 1 \rangle + \frac{(-n)(-n-1)}{2} \langle -1 \rangle \end{aligned}$$

Each point of the plane  $\mathbb{R}^2$  we will denote by  $\langle 0 \rangle$ . Later each triangle  $\langle \pm n \rangle$ , where  $n \in \mathbb{N}$  we will denote by the symbol  $\langle n \rangle$ , where  $n \in \mathbb{Z}$ .

The symbol  $-\langle n \rangle$  denotes the triangle, which lying on the triangle  $\langle n \rangle$  gives the empty set. The more precision geometric interpretation of the elements  $\pm\langle n \rangle$  will be given in subsection 1.3.

The set  $\mathbb{N}_2 = \{\pm\langle n \rangle; n \in \mathbb{Z}\}$  is the subset of the ring

$$\mathbb{P}_2(\mathbb{Z}) = \{(x, y) = x\langle 1 \rangle + y\langle -1 \rangle, x, y \in \mathbb{Z}\}$$

with addition

$$\forall x_1, x_2, y_1, y_2 \in \mathbb{Z} \quad (x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$$

and commutative multiplication

$$(x_1, y_1) \cdot (x_2, y_2) = (x_1x_2 + y_1y_2, x_1y_2 + x_2y_1)$$

and the neutral element  $(0, 0) = \langle 0 \rangle$  of addition and neutral element  $(1, 0) = \langle 1 \rangle$  of multiplication. So each element  $\langle n \rangle \in \mathbb{N}_2$  we can write in the form  $\langle n \rangle = (\frac{n(n+1)}{2}, \frac{n(n+1)}{2})$ .

Because we will be looking for connection between the natural or integer numbers and the elements of the set  $\mathbb{N}_2$  so the element of the set  $\mathbb{N}_2$  we will be calling them interchangeably the triangles or the numbers. It is easy to see that

$$\langle -1 \rangle^2 = \langle 1 \rangle$$

$$\langle 1 \rangle \cdot \langle -1 \rangle = \langle -1 \rangle \cdot \langle 1 \rangle = \langle -1 \rangle$$

and  $\forall n, m \in \mathbb{Z}$

$$\begin{aligned} \langle n \rangle \cdot \langle m \rangle &= \langle nm \rangle \\ &= \left( \frac{n(n+1)}{2} \frac{m(m+1)}{2} + \frac{n(n-1)}{2} \frac{m(m-1)}{2}, \right. \\ &\quad \left. \frac{n(n+1)}{2} \frac{m(m-1)}{2} + \frac{n(n-1)}{2} \frac{m(m+1)}{2} \right) \\ &= \frac{n(n+1)}{2} \left( \frac{m(m+1)}{2}, \frac{m(m-1)}{2} \right) \\ &\quad + \frac{n(n-1)}{2} \left( \frac{m(m-1)}{2}, \frac{m(m+1)}{2} \right) \\ &= \left( \frac{nm(nm+1)}{2}, \frac{nm(nm-1)}{2} \right) \\ &= \langle nm \rangle \end{aligned} \tag{2}$$

The sequence of the equations (2) is the answer the question, why  $\frac{n(n+1)}{2}$  triangles  $\langle m \rangle$  and  $\frac{n(n-1)}{2}$  triangles  $\langle -m \rangle$  give the triangle  $\langle nm \rangle$  with sides  $n$  times bigger than sides of the triangle  $\langle m \rangle$ .

Let us transform

$$\langle n \rangle = \frac{n^2 + n}{2} \langle 1 \rangle + \frac{n^2 - n}{2} \langle -1 \rangle = n^2 \frac{\langle 1 \rangle + \langle -1 \rangle}{2} + n \frac{\langle 1 \rangle - \langle -1 \rangle}{2}$$

It is easy to show that elements  $\frac{\langle 1 \rangle + \langle -1 \rangle}{2} = A_2$  and  $\frac{\langle 1 \rangle - \langle -1 \rangle}{2} = A_1$  are orthogonal. Let us denote  $A_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ ,  $A_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ . Then

$$\langle n \rangle = n^2 A_2 + n A_1 = \begin{pmatrix} n^2 \\ n \end{pmatrix} \quad (3)$$

we can call the orthogonal form of the triangle  $\langle n \rangle$ . Using the linear combination of the elements of the set  $\mathbb{N}_2$  we can write different geometric figures (Fig. 2, 3).

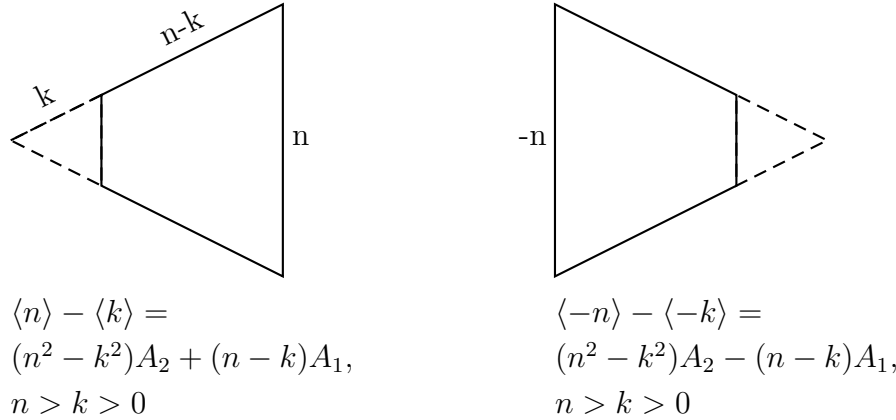
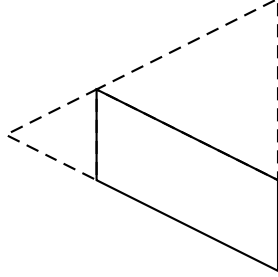


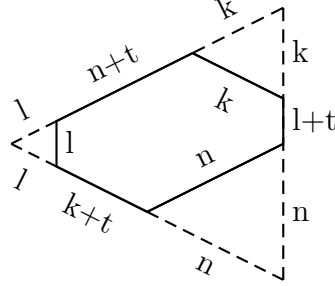
Fig. 2.

From the description of (Fig. 2, 3) we can see that the coefficient of  $A_2$  means the number of all triangles  $\langle \pm 1 \rangle$  in the given figure. The second coefficient of  $A_1$  equals to the difference in the lengths of the parallel sides (the unit side length equals to the length of the parallel side of the triangle  $\langle 1 \rangle$ ). So if the triangle  $\langle 1 \rangle$  is equilateral or not, the length of each side is equal to 1.



$$\langle n+k \rangle - \langle n \rangle - \langle k \rangle = 2nkA_2,$$

$$n > k > 0$$



$$\langle n+k+l+t \rangle - \langle n \rangle - \langle k \rangle - \langle l \rangle = xA_2 + tA_1,$$

$$n, k, l, t \in \mathbb{N}$$

Fig. 3.

Let us put the addition closed in  $\mathbb{N}_2$ :

$$\forall n, k, l \in \mathbb{Z}$$

$$\langle n+k+l \rangle = \langle n+k \rangle + \langle n+l \rangle + \langle k+l \rangle - \langle n \rangle - \langle k \rangle - \langle l \rangle \quad (4)$$

The operation is well-defined. This results from orthogonal form of  $\langle n \rangle$  and the truth of the below condition:

$$\forall n, k, l \in \mathbb{Z} \quad \forall i = 1, 2$$

$$(n+k+l)^i = (n+k)^i + (n+l)^i + (k+l)^i - (n)^i - (k)^i - (l)^i.$$

If we assume  $nk \neq 0$ ,  $l = 0$  in (4) then the equation (4) takes the form

$$\langle n+k \rangle = \langle n+k \rangle + \langle n \rangle + \langle k \rangle - \langle n \rangle - \langle k \rangle$$

i.e.  $\langle n+k \rangle = \langle n+k \rangle$

So we don't get the new element.

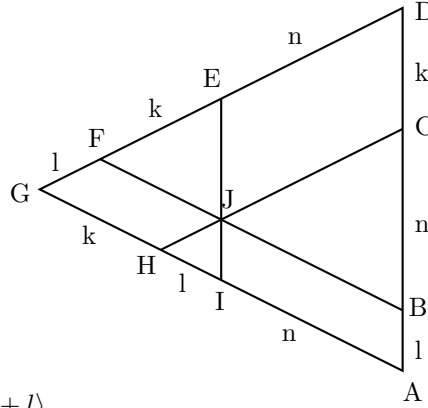
Using elements  $\langle 1 \rangle$ ,  $\langle 2 \rangle$  and operation (4) we can obtain each element  $\langle n+2 \rangle$ , for  $n \in \mathbb{N}$ . Indeed

$$\langle 3 \rangle = \langle 1+1+1 \rangle = 3\langle 2 \rangle - 3\langle 1 \rangle,$$

$$\langle n + 2 \rangle = \langle n + 1 + 1 \rangle = 2\langle n + 1 \rangle + \langle 2 \rangle - \langle n \rangle - 2\langle 1 \rangle$$

Here we have the analogy to the definition of natural numbers by using the element zero and the notion of successor. To define elements of the set  $\{\langle n \rangle; n > 0\}$  we have to have the primitive concept of  $\langle 2 \rangle$  and  $\langle 1 \rangle$  and the notion of successor  $\langle Sn \rangle = \langle n + 1 \rangle$  for  $n > 1$ .

The operation (4) has the simple geometric interpretation for  $n, k, l > 0$  (Fig. 4).



$$\triangle ADG = \langle n + k + l \rangle$$

$$\triangle BDF = \langle n + k \rangle$$

$$\triangle BCJ = \langle n \rangle$$

$$\triangle ACH = \langle n + l \rangle$$

$$\triangle EFJ = \langle k \rangle$$

Fig. 4.

$$\triangle EGI = \langle k + l \rangle$$

$$\triangle HIJ = \langle l \rangle$$

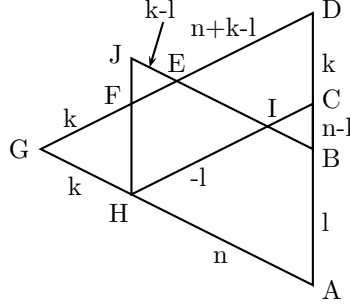
The (Fig. 5) gives the geometric interpretation of equation (4) with the change  $l > 0$  on  $-l < 0$ .

$$\langle n + k - l \rangle = \langle n + k \rangle + \langle n - l \rangle + \langle k - l \rangle - \langle n \rangle - \langle k \rangle - \langle -l \rangle$$

It is easy to show that

$$\langle n \rangle = \frac{n(n-1)}{2} \langle 2 \rangle - n(n-2) \langle 1 \rangle \quad (5)$$

We can call the equation (5) the arithmetic form of the number  $\langle n \rangle$  for  $n > 0$ , because it is an equivalent of the equation  $n = n \cdot 1$  in the set  $\mathbb{N}$ .



$$\triangle BDE = \langle n + k - l \rangle$$

$$\triangle ADG = \langle n + k \rangle$$

$$\triangle FGH = \langle k \rangle$$

$$\triangle ACH = \langle n \rangle$$

$$\triangle HIJ = \langle -l \rangle$$

$$\triangle BCI = \langle n - l \rangle, \quad n - l > 0$$

$$\triangle EJI = \langle k - l \rangle, \quad k - l < 0$$

Fig. 5.

Let us derive the formula of four and more elements:

$$\begin{aligned} \langle n + k + l + m \rangle &= \langle n + k + (l + m) \rangle = \\ &= \langle n + k \rangle + \langle k + l + m \rangle + \langle n + l + m \rangle - \langle n \rangle - \langle k \rangle - \langle l + m \rangle \\ &= \langle n + k \rangle + \langle n + l \rangle + \langle n + m \rangle + \langle k + l \rangle + \langle k + m \rangle + \langle l + m \rangle \\ &\quad - 2\langle n \rangle - 2\langle k \rangle - 2\langle l \rangle - 2\langle m \rangle. \end{aligned}$$

$$\langle m_1 + m_2 + \cdots + m_n \rangle = \sum_{\substack{i,j \\ i \neq j}} \langle m_i + m_j \rangle - (n - 2) \sum_{i=1}^n \langle m_i \rangle.$$

then for  $n > 2$

$$\langle nm \rangle = \langle \underbrace{m + m + \cdots + m}_n \rangle = \frac{n(n-1)}{2} \langle 2m \rangle - (n-2)n \langle m \rangle.$$

We can set the certain kind of the multiplication for  $n > 2$

$$\langle n * m \rangle = \langle \underbrace{m + m + \cdots + m}_n \rangle = \frac{n(n-1)}{2} \langle 2m \rangle - (n-2)n \langle m \rangle = \langle nm \rangle. \quad (6)$$

Let us notice there exists the product  $\langle n * 2 \rangle$  for  $n > 2$  but not exists  $\langle 2 * n \rangle$ . So we can call the number  $\langle 2p \rangle$ , where  $p$  is the prime number, the one-sided

composite number. If we set isomorphism  $f : \langle n \rangle \mapsto n, \forall n \in \mathbb{N}$  with the operation  $*$  we will get the new look on the set  $\mathbb{N}$ . Each natural number  $n > 2$  is built from numbers 1 and 2 but on other hand the number 2 loose the place between the composite numbers.

If we put  $l = n + k$  in the equation (4) we will get

$$\begin{aligned} \langle n + k + (n + k) \rangle &= \langle n + k \rangle + \langle n + 2k \rangle + \langle 2n + k \rangle \\ &\quad - \langle n \rangle - \langle k \rangle - \langle n + k \rangle \\ &= \langle n + 2k \rangle + \langle 2n + k \rangle - \langle n \rangle - \langle k \rangle \end{aligned} \quad (7)$$

The equation (7) is a particular case of the following theorem.

**Theorem 1.** *The natural number  $z$  is composite if and only if the exist smaller than  $z$  natural positive numbers  $a, b, c, d$  which satisfying the following equation*

$$\langle z \rangle = \langle a \rangle + \langle b \rangle - \langle c \rangle - \langle d \rangle. \quad (8)$$

*Proof.* Let us suppose that  $z$  is composite number. Then there exist positive  $x, y, n, m \in \mathbb{N}$  and

$$z = (x + y)(m + n).$$

For  $i = 1, 2$  there is the following condition

$$z^i = (xm + ym + xn)^i + (ym + xn + yn)i - (ym)^i - (xn)^i, \quad (9)$$

which gives us (8). On the other hand, let us suppose the exist smaller than  $z$  natural positive numbers  $a, b, c, d$  satisfying the following system of equations

$$\begin{cases} z &= a + b - c - d \\ z^2 &= a^2 + b^2 - c^2 - d^2. \end{cases}$$

If we put  $a = z - b + c + d$  to the second equation, we will obtain

$$z = \frac{(b - c)(b - d)}{b - c - d}.$$

We can see that  $b - c - d > 0$ . Otherwise, it would be  $a \geq z$  against the supposing. Let us denote  $b - c - d = t$ . Then

$$z = \frac{(t + d)(t + c)}{t} = t + c + d + \frac{dc}{t},$$



and it must be  $t = t_1 t_2$ ,  $c = s_1 t_1$ ,  $d = s_2 t_2$ , where  $t_1, t_2, s_1, s_2 \in \mathbb{N}$ . So

$$z = t_1 t_2 + s_1 t_1 + s_2 t_2 + s_1 s_2 = (t_1 + s_1)(t_2 + s_2)$$

and  $z$  is the composite number.  $\square$

**Remark.** *The equation (9) is a special case of the solution of the Tarry-Escott problem for  $k=2$  [1, 2].*

## 1.2 Generalization of the equation (4)


We can generalize the equation (4). Namely

$$\begin{aligned} \forall n, k, l, t \in \mathbb{Z} \\ \langle n + k + l + t \rangle &= \langle n + k + t \rangle + \langle n + l + t \rangle + \langle k + l + t \rangle \\ &\quad - \langle n + t \rangle - \langle k + t \rangle - \langle l + t \rangle + \langle t \rangle \end{aligned} \quad (10)$$

The equation (10) is true because the below equation holds for  $i = 2, 1$  and  $i = 0$ .

$$\begin{aligned} (n + k + l + t)^i &= (n + k + t)^i + (n + l + t)^i + (k + l + t)^i \\ &\quad - (n + t)^i - (k + t)^i - (l + t)^i + t^i. \end{aligned}$$

We will use the equation (10) to describe the following geometric figure

 . Because it is difficult to understand this description we approach to it in some steps ((Fig. 6-9),  $n, k, l, m > 0$  ).

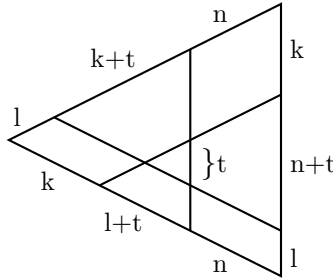


Fig. 6. Interpretation of the equation (10).

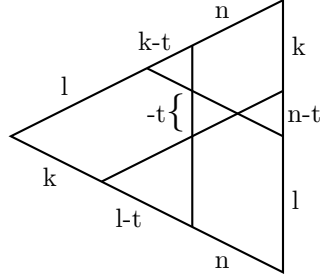
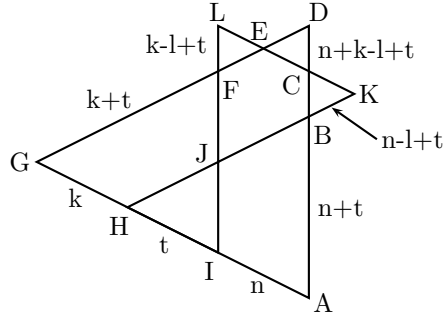


Fig. 7. Interpretation of the equation (10) with the left side equal to  $\langle n + k + l - t \rangle$ .



$$HK = IL = -l$$

$$\triangle HIJ = \langle t \rangle$$

$$\triangle ABH = \langle n + t \rangle$$

$$\triangle FGI = \langle k + t \rangle$$

$$\triangle JKL = \langle -l + t \rangle$$

$$\triangle ADG = \langle n + k + t \rangle$$

$$\triangle BCK = \langle n - l + t \rangle, \quad n - l + t < 0$$

$$\triangle EFL = \langle k - l + t \rangle, \quad k - l + t < 0$$

$$\triangle CDE = \langle n + k - l + t \rangle.$$

Fig. 8. Interpretation of the equation (10) with the left side equal to  $\langle n + k - l + t \rangle$ .

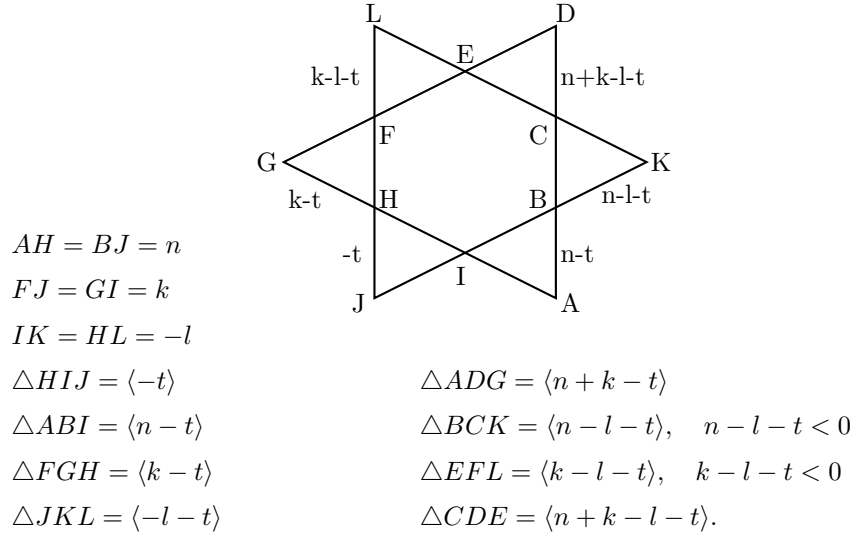


Fig. 9. Interpretation of the equation (10) with the left side equal to  $\langle n + k - l - t \rangle$ .

If we put  $n = k = l = 2t$  in the below equation

$$\begin{aligned}
 \langle n + k - l - t \rangle &= \langle n + k - t \rangle + \langle n - l - t \rangle + \langle k - l - t \rangle \\
 &\quad - \langle n - t \rangle - \langle k - t \rangle - \langle -l - t \rangle + \langle -t \rangle
 \end{aligned}$$


then we will get

$$\langle t \rangle = \langle 3t \rangle + \langle -t \rangle + \langle -t \rangle - \langle t \rangle - \langle t \rangle - \langle -3t \rangle + \langle -t \rangle$$

and

$$3\langle t \rangle + \langle -3t \rangle = 3\langle -t \rangle + \langle 3t \rangle \quad (11)$$

The equation (11) is true for each integer  $t$ . We can get it directly from the

figure (9). It is enough to see that the figure  we can build from the elements of the left or the right side of the equation (11).

Using the formula (10) we can get three following equation.

$$\begin{aligned}\langle n+k+t \rangle &= \langle n+k+(-l)+(l+t) \rangle \\ &= \langle n+k+l+t \rangle + \langle n+t \rangle + \langle k+t \rangle \\ &\quad - \langle n+l+t \rangle - \langle k+l+t \rangle - \langle t \rangle + \langle l+t \rangle\end{aligned}$$

$$\begin{aligned}\langle n+t \rangle &= \langle n+(-k)+(-l)+(k+l+t) \rangle \\ &= \langle n+l+t \rangle + \langle n+k+t \rangle + \langle t \rangle \\ &\quad - \langle n+k+l+t \rangle - \langle l+t \rangle - \langle k+t \rangle + \langle k+l+t \rangle\end{aligned}$$

$$\begin{aligned}\langle t \rangle &= \langle (-n)+(-k)+(-l)+(n+k+l+t) \rangle \\ &= \langle l+t \rangle + \langle k+t \rangle + \langle l+t \rangle \\ &\quad - \langle k+l+t \rangle - \langle n+l+t \rangle - \langle n+k+t \rangle + \langle n+k+l+t \rangle\end{aligned}$$

They are equivalent to difference in additive group  $G$ . That means, if  $n, k, l$  are elements of the group  $G$  and satisfy the equation  $l = n + k$  then there exists an element  $(-k) \in G$  fulfilling  $n = l + (-k)$ .

You can show that we can't get similar equations from the formula (4).

We remember, we can replace  $\langle n \rangle$  by  $n^i$  for  $i = 2, 1, 0$  in the equation (10). In order to distinguish when  $\langle n \rangle$  is  $n^i$  for  $i = 2, 1$  and when  $n^i$  for  $i = 2, 1, 0$ , we will denote the last one by  $\langle n \rangle_0$ . So the equation (10) is true for  $\langle n \rangle_0$ . From (10) we have the next equation

$$\langle 4 \rangle_0 = \langle 1+1+1+1 \rangle_0 = 3\langle 3 \rangle_0 - 3\langle 2 \rangle_0 + \langle 1 \rangle_0$$

It inspirit us to find

$$\langle n \rangle_0 = \frac{(n-1)(n-2)}{2} \langle 3 \rangle_0 - 2 \frac{(n-1)(n-3)}{2} \langle 2 \rangle_0 + \frac{(n-2)(n-3)}{2} \langle 1 \rangle_0$$

and more general equation

$$\begin{aligned}\langle n \rangle_0 &= \frac{(n-k)(n-(k-1))}{2} \langle k+1 \rangle_0 - 2 \frac{(n-(k+1))(n-(k-1))}{2} \langle k \rangle_0 \\ &\quad + \frac{(n-k)(n-(k+1))}{2} \langle k-1 \rangle_0\end{aligned}\tag{12}$$

From (12) we can get for  $k = 1$  the arithmetic form of the element  $\langle n \rangle_0$

$$\langle n \rangle_0 = \frac{n(n-1)}{2} \langle 2 \rangle_0 - 2 \frac{(n-2)n}{2} \langle 1 \rangle_0 + \frac{(n-2)(n-1)}{2} \langle 0 \rangle_0 \quad (13)$$

and for  $k = 0$  the geometric form

$$\langle n \rangle_0 = \frac{n(n+1)}{2} \langle 1 \rangle_0 - 2 \frac{(n-1)(n+1)}{2} \langle 0 \rangle_0 + \frac{n(n-1)}{2} \langle -1 \rangle_0 \quad (14)$$

From (14) we have

$$\langle n \rangle_0 = n^2 \frac{\langle 1 \rangle_0 - 2 \langle 0 \rangle_0 + \langle -1 \rangle_0}{2} + n \frac{\langle 1 \rangle_0 - \langle -1 \rangle_0}{2} + 1 \langle 0 \rangle_0$$

If we put  $\frac{\langle 1 \rangle_0 - 2 \langle 0 \rangle_0 + \langle -1 \rangle_0}{2} = A_2$ ,  $\frac{\langle 1 \rangle_0 - \langle -1 \rangle_0}{2} = A_1$ ,  $\langle 0 \rangle_0 = A_0$

we will obtain  $\langle 1 \rangle_0 = A_2 + A_1 + A_0$ ,  $\langle -1 \rangle_0 = A_2 - A_1 + A_0$ .

From the assumption that the condition

$$\forall n, m \in \mathbb{Z} \quad \langle n \rangle_0 \cdot \langle m \rangle_0 = \langle nm \rangle_0$$

is fulfilled we have an orthogonality of the elements  $A_2, A_1, A_0$ . If we write

$$A_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad A_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad A_0 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix},$$

we will get

$$\langle n \rangle_0 = n^2 A_2 + n A_1 + 1 A_0 = \begin{pmatrix} n^2 \\ n \\ 1 \end{pmatrix} \quad (15)$$

The set  $\mathbb{N}_{20} = \{\pm \langle n \rangle_0; n \in \mathbb{Z}\}$  is the subset of the ring

$$\mathbb{P}_3 = \{x A_2 + y A_1 + z A_0; x, y, z \in \mathbb{Z}\}.$$

Comparing (15) with the element  $\langle n \rangle = n^2 A_2 + n A_1 = \begin{pmatrix} n^2 \\ n \\ 0 \end{pmatrix}$  you can see that the element  $\langle n \rangle$  we can get from the element  $\langle n \rangle_0$  by cutting the last coordinate. So if element  $\langle n \rangle_0$  satisfies any equation then element  $\langle n \rangle$  satisfies too. Now it can be concluded that we can obtain the arithmetic and geometric form of the number  $\langle n \rangle$  from the equations (13) and (14) respectively.

### 1.3 Geometric interpretation of elements from $\mathbb{N}_{20}$

From the equation (10) we can get more precision geometric interpretation of elements of the set  $\mathbb{N}_{20}$ , and therefore also elements of the set  $\mathbb{N}_2$ . Earlier would be better to show the geometric interpretation of integer numbers. We can notice that the usual addition of two numbers from the set  $\mathbb{N}_1 = \mathbb{Z}$  is equivalent to the operation (4) in the set  $\mathbb{N}_2$ . Furthermore the generalization of adding two integer numbers to elements of the set  $\mathbb{N}_{10} = \{\langle n \rangle_{10} = nA_1 + 1A_0 = \binom{n}{1}; n \in \mathbb{Z}\}$  is the following operation

$$\forall n, k, t \in \mathbb{Z} \quad \langle n + k + t \rangle_{10} = \langle n + t \rangle_{10} + \langle k + t \rangle_{10} - \langle t \rangle_{10},$$

So like the generalization of the operation (4) is the operation (10). In turn the following equation

$$\langle n \rangle_{10} = (n - k)\langle k + 1 \rangle_{10} - (n - k - 1)\langle k \rangle_{10}, \quad (16)$$

is equivalent to the equation (12). From (16) we are getting for  $n = 0$  and  $n = k + 2$  respectively

$$\langle n \rangle_{10} = n\langle 1 \rangle_{10} - (n - 1)\langle 0 \rangle_{10} \quad (17)$$

$$\langle k + 2 \rangle_{10} = 2\langle k + 1 \rangle_{10} - \langle k \rangle_{10} \quad (18)$$

Let us put several findings. We will denote existing figures in black and not existing ones in red. If we put black figure on red one we will get the empty set which we will mark in green. Let us denote the fixed double-side closed segment  $\bullet \text{---} \bullet$  by  $\langle 1 \rangle_{10}$ . The  $n$  times longer closed segment we will denote by  $\langle n \rangle_{10}$  for  $n > 0$ . The point  $\bullet$  we will denote by  $\langle 0 \rangle_{10}$ . (Fig.10-12) show geometric interpretation of the elements from the set  $\mathbb{N}_{10}$ .

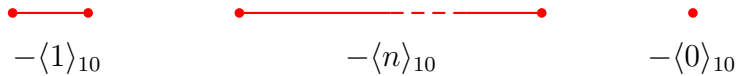


Fig. 10.

If we put  $k = -2$  in (18) we will get

$$\langle 0 \rangle_{10} = 2\langle -1 \rangle_{10} - \langle -2 \rangle_{10}$$

$$\langle n \rangle_{10} = n\langle 1 \rangle_{10} - (n-1)\langle 0 \rangle_{10}$$

Fig. 11. Fig. 11. Interpretation of the equation (17)

$$\langle -2 \rangle_{10} = -2\langle 1 \rangle_{10} + 3\langle 0 \rangle_{10}$$

Fig. 12. Green ends of the segment mean the open segment


And after transformation we will get  $\langle -2 \rangle_{10}$  in another way (Fig. 13).

$$\langle -2 \rangle_{10} = 2\langle -1 \rangle_{10} - \langle 0 \rangle_{10}$$

Fig. 13.

Let us notice that  $\langle -2 \rangle_{10}$  differs from  $-\langle 2 \rangle_{10}$  (Fig14). Similarly  $\langle -n \rangle_{10} \neq -\langle n \rangle_{10}$  for all  $n \in \mathbb{N}$ .

Now we can go to demonstrate geometric interpretation of the element  $\langle n \rangle_0$

from  $\mathbb{N}_{20}$ . Let us denote the fixed triangle  with sides and vertexes

by  $\langle 1 \rangle_0$ . The interior of this triangle should be black but we wanted to mark that sides and vertexes belong to this triangle, so we colored it in gray. The black triangle congruent to  $\langle 1 \rangle_0$  with vertexes and  $n$  times longer sides we will denote by  $\langle n \rangle_0$ . The point  $\bullet$  on the plane we will denote by  $\langle 0 \rangle_0$ . Then the closed red triangle we will denote by  $-\langle n \rangle_0$ .

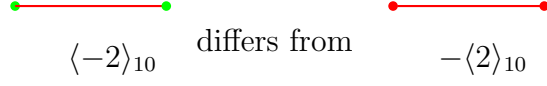


Fig. 14.

From the equation (13) for  $n = -1$  we have

$$\langle -1 \rangle_0 = \langle 2 \rangle_0 - 3\langle 1 \rangle_0 + 3\langle 0 \rangle_0$$

(Fig. 15) shows each step of a construction of this element.

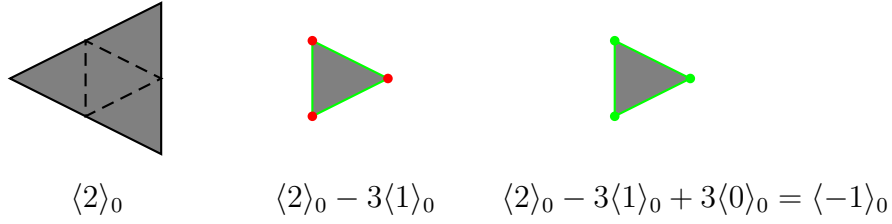


Fig. 15.

From the equation (12) it is easy to get  $\langle -n \rangle_0 = \langle 2n \rangle_0 - 3\langle n \rangle_0 + 3\langle 0 \rangle_0$ . Therefore from (Fig. 15) results that the triangle  $\langle -n \rangle_0$  for  $n > 0$  is the opened triangle.

Let us show once more example of the geometric interpretation (Fig. 16) of the following equation  $6\langle 1 \rangle_0 - 8\langle 0 \rangle_0 + 3\langle -1 \rangle_0 = \langle 3 \rangle_0$ .

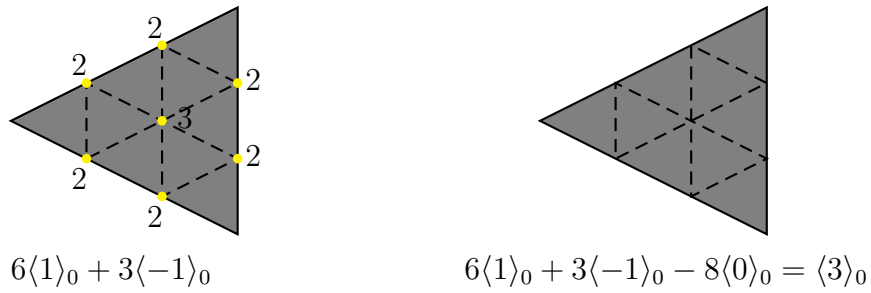


Fig. 16.



Numbers 2 and 3 staying near yellow points mean an overlapping two or three points and necessity of taking away one  $\langle 0 \rangle_0$  or two  $\langle 0 \rangle_0$  in these points.

#### 1.4 Geometric interpretation of $\langle n \rangle - \langle k \rangle$

We can present each difference  $n - k$  for  $n, k \in \mathbb{N}$  as the sum of two vectors with opponent senses lying on one axis. We can say that difference  $n - k$  is a segment with the zero inside this segment (Fig. 17). We will write it

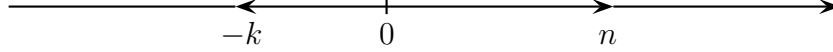


Fig. 17.  $n - k = (n, k)$

down in the form of the ordered pair:  $n - k = (n, k)$ . It is easy to check the equation  $(n, k) = (n + x, k + x)$  for all  $x \in \mathbb{N}$ . Similarly, we would like the triangle with the zero inside to be the expression  $(n, k, l)$  (Fig. 18) depending on three independent values  $n, k, l \in \mathbb{N}$  and satisfying the condition:

$$\forall x \in \mathbb{N} \quad (n + x, k + x, l + x) = (n, k, l).$$

So we need to describe each of the triangles  $\langle n \rangle$  and  $\langle -n \rangle$  in three ways. These ways would be an equivalent of two senses of vector on the line. We will first seek a solution of this problem in the set of the hypercomplex numbers [3, 4]:

$$\mathbb{T} = \{x + ye + zi + vj; \ x, y, z, v \in \mathbb{R}\}, \quad \text{where } e = \langle -1 \rangle,$$

with the multiplication

$$i^2 = j^2 = -e^2 = -1, \quad ij = ji = -e, \quad ej = je = i, \quad ei = ie = j.$$

If we describe the particular parts of the triangle like in (Fig. 19), where

$$\begin{aligned} \varepsilon &= \frac{1 - \sqrt{3}i}{2}, & \varepsilon^* &= \frac{-1 - \sqrt{3}i}{2}, \\ \langle \varepsilon \rangle &= \varepsilon^2 A_2 + \varepsilon A_1 = \varepsilon^* A_2 + \varepsilon A_1, & \langle \varepsilon^* \rangle &= \varepsilon^{*2} A_2 + \varepsilon^* A_1 = \varepsilon A_2 + \varepsilon^* A_1, \end{aligned}$$

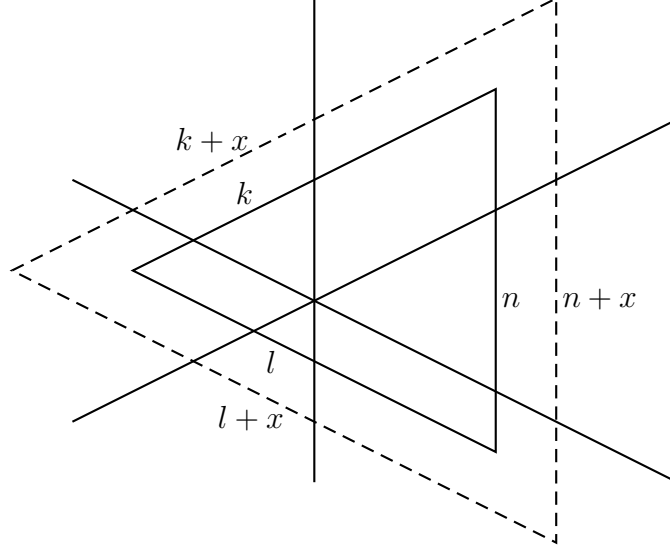


Fig. 18.  $(n, k, l) = (n + x, k + x, l + x)$

we will get

$$(n, k, l) = (n + \varepsilon k + \varepsilon^* l)^2 A_2 + (n + \varepsilon k + \varepsilon^* l) A_1.$$

It seems the set  $\mathbb{T}$  is too rich for description of the triangle because the set  $\mathbb{T}$  has  $n^2$  nth roots of the unity. Four square roots is too little to describe and nine cube roots is too much. For example, we have the geometric interpretation of the element  $\langle \varepsilon \rangle$  but we have not the geometric interpretation of the element  $\langle \varepsilon_j \rangle = \langle \frac{1-\sqrt{3}j}{2} \rangle$ .

In  $\mathbb{P}_2(\mathbb{Z})$  we will get simpler description of  $(n, k, l)$ . If we describe the unit triangles as in (Fig. 20) we will obtain

$$(n, k, l) = (n - l)(n - 2k + l)A_2 + (n - 2k + l)A_1 = \langle n - k \rangle - \langle k - l \rangle.$$

Because  $(n, k, l) = (n - l, k - l, 0)$  we can investigate only elements of the form  $(n, k, 0)$ . Then it is easy to check the truth of the addition and the multiplication.

$$\begin{aligned} (n_1 + n_2 + n_3, k_1 + k_2 + k_3, 0) = \\ = (n_1 + n_2, k_1 + k_2, 0) + (n_2 + n_3, k_2 + k_3, 0) + (n_1 + n_3, k_1 + k_3, 0) - \\ - (n_1, k_1, 0) - (n_2, k_2, 0) - (n_3, k_3, 0) \end{aligned}$$

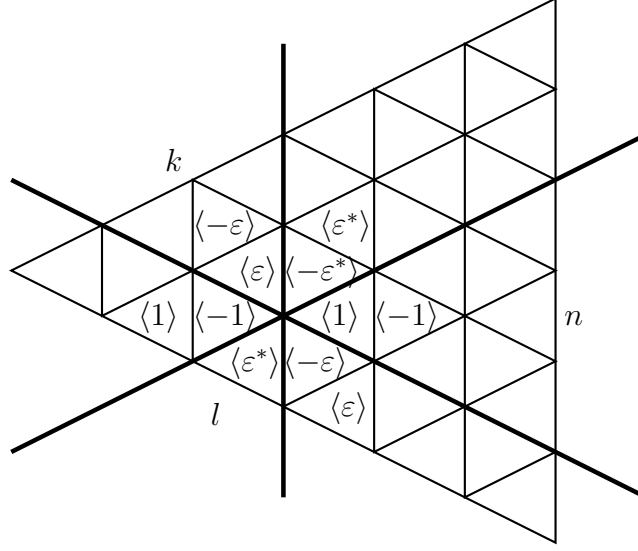


Fig. 19.  $(n, k, l) = \langle n + \varepsilon k + \varepsilon^* l \rangle = (n + x, k + x, l + x)$

$$(n_1, k_1, 0)(n_2, k_2, 0) = (n_1 n_2, n_1 k_2 + n_2 k_1 - 2k_1 k_2, 0)$$

On (Fig. 21) we can see tree senses of the triangle  $\langle n \rangle$  and tree senses of the triangle  $\langle -n \rangle$ .

On (Fig.22, 23) we can trace the stages of transition from  $(3, 1, 0)$  to  $(0, -2, -3)$ .

The problem of finding a proper description of the triangle with the zero inside was only signaled and requires further study.

## 1.5 Problems to solve

### 1.5.1 Problem 1

Let us consider the equation (4) for the concrete numbers

$$\begin{aligned} \langle 4 \rangle &= \langle 1 + 1 + 2 \rangle = 2\langle 3 \rangle + \langle 2 \rangle - \langle 2 \rangle - 2\langle 1 \rangle \\ &= 2\langle 3 \rangle - 2\langle 1 \rangle \end{aligned}$$

From the arithmetic point of view the equation

$$\langle 4 \rangle = 2\langle 3 \rangle - 2\langle 1 \rangle$$

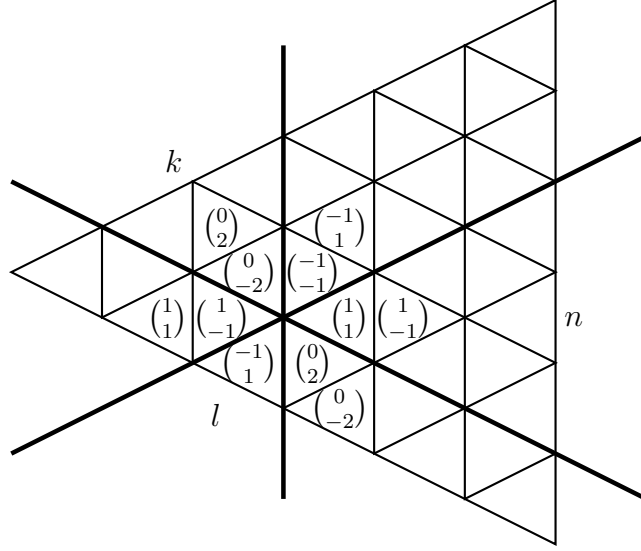


Fig. 20.  $(n, k, l) = \langle n - k \rangle - \langle k - l \rangle = (n + x, k + x, l + x)$

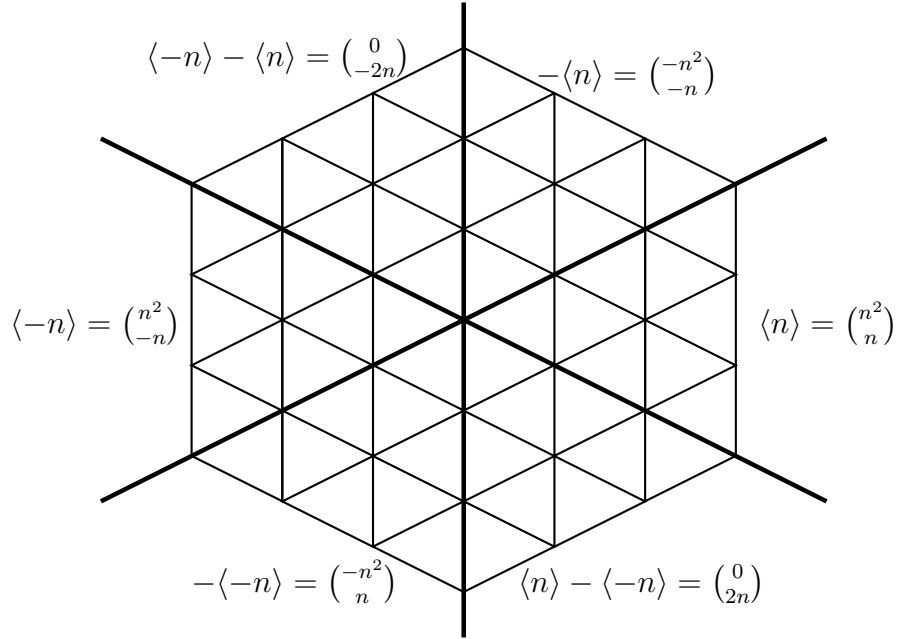
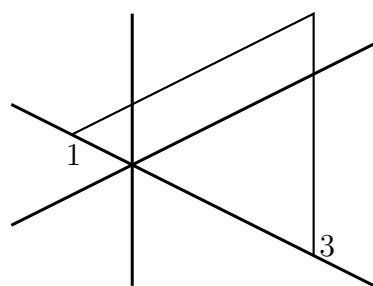
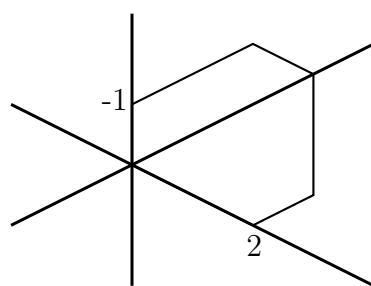


Fig. 21.

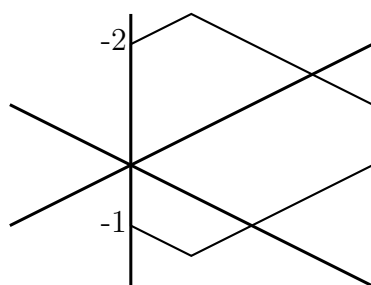


$$(3, 1, 0)$$



$$(3 - 1, 1 - 1, 0 - 1) = (2, 0, -1)$$

Fig. 22.



$$(2 - 1, 0 - 1, -1 - 1) = (1, -1, -2) \quad (1 - 1, -1 - 1, -2 - 1) = (0, -2, -3)$$

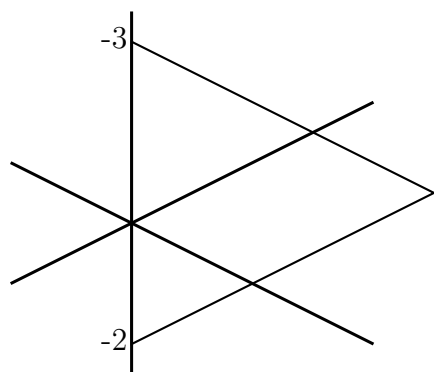


Fig. 23.

is true. But it easy to see that we cannot build the triangle  $\langle 4 \rangle$  using only two black triangles  $\langle 3 \rangle$  and two red triangles  $-\langle 1 \rangle$ . We need the triangles  $\langle 2 \rangle$  and  $-\langle 2 \rangle$  too. They are not reducible to the empty set because they do not lie on one another. (Fig. 24).

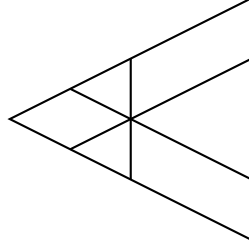


Fig. 24.

If we will divide the triangle  $\langle 3 \rangle$  on the smaller ones, that means we will write

$$\langle 3 \rangle = 3\langle 2 \rangle - 3\langle 1 \rangle$$

then

$$\begin{aligned} \langle 4 \rangle &= \langle 3 \rangle + \langle 3 \rangle + \langle 2 \rangle - \langle 2 \rangle - 2\langle 1 \rangle \\ &= \underline{3\langle 2 \rangle} - 3\langle 1 \rangle + \underline{3\langle 2 \rangle} - 3\langle 1 \rangle + \langle 2 \rangle - \langle 2 \rangle - 2\langle 1 \rangle \end{aligned}$$

One triangle of triangles  $\underline{\langle 2 \rangle}$ , one triangle of triangles  $\underline{\underline{\langle 2 \rangle}}$  and triangle  $-\langle 2 \rangle$  are reducing to one triangle  $\langle 2 \rangle$  (Fig. 25). So

$$\langle 4 \rangle = 6\langle 2 \rangle - 8\langle 1 \rangle = 2\langle 3 \rangle - 2\langle 1 \rangle.$$

If we write the pattern (4) in the form of matrix

$$\langle n + k + l \rangle = \begin{pmatrix} n + k & k + l & l + n \\ n & k & l \end{pmatrix}$$

we can see that the triangle  $-\langle k \rangle$  can reduce with the common part of  $\langle n + k \rangle$  and  $\langle k + l \rangle$ , the triangle  $-\langle l \rangle$  can reduce with the common part of  $\langle k + l \rangle$  and  $\langle l + n \rangle$  and the triangle  $-\langle n \rangle$  with the common part of  $\langle l + n \rangle$  and  $\langle n + k \rangle$ . The question arises whether there is operation which would reduce at least tree disjoint triangles to the element  $\langle 0 \rangle$  or to the empty set. This question seems to be a key to further study the sets  $\mathbb{N}_2$  and  $\mathbb{N}_{20}$ .

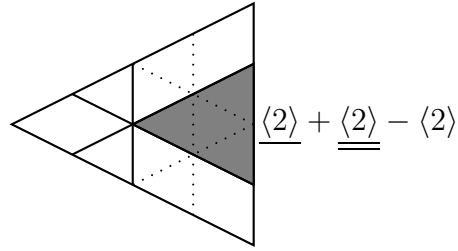


Fig. 25.

### 1.5.2 Problem 2

We can extend the set  $\mathbb{N}_2$  to the set

$$\mathbb{Q}_2 = \{\pm \langle q \rangle = \pm(q^2 A_2 + q A_1); q \in \mathbb{Q}\} \subset \mathbb{P}_2(\mathbb{Q}),$$

where  $\mathbb{Q}$  is the set of rational numbers and  $\mathbb{P}_2(\mathbb{Q})$  is the ring with the same operations as in the ring  $\mathbb{P}_2(\mathbb{Z})$ . Let us fulfill the triangle  $\langle 1 \rangle$  by the triangles  $\langle -\frac{1}{2^n} \rangle$  (Fig. 26). If we add up the values  $(-\frac{1}{2^n})^2 A_2 + (-\frac{1}{2^n}) A_1$  we will get

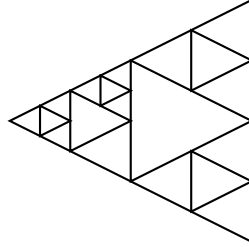


Fig. 26.

$$\begin{aligned}
\langle -\frac{1}{2} \rangle + 3\langle -\frac{1}{2^2} \rangle + 3^2\langle -\frac{1}{2^3} \rangle + \dots &= \\
&= \lim_{n \rightarrow \infty} \left( \frac{1}{2^2} + \frac{3}{2^4} + \frac{3^2}{2^6} + \dots + \frac{3^{n-1}}{2^{2n}} \right) A_2 \\
&\quad - \lim_{n \rightarrow \infty} \left( \frac{1}{2} + \frac{3}{2^2} + \frac{3^2}{2^3} + \dots + \frac{3^{n-1}}{2^n} \right) A_1 \\
&= \lim_{n \rightarrow \infty} \frac{1}{4} \cdot \frac{1 - \left(\frac{3}{4}\right)^n}{1 - \frac{3}{4}} A_2 - \lim_{n \rightarrow \infty} \frac{1}{2} \cdot \frac{1 - \left(\frac{3}{2}\right)^n}{1 - \frac{3}{2}} A_1 \\
&= \lim_{n \rightarrow \infty} \left[ 1 - \left(\frac{3}{4}\right)^n \right] A_2 + \lim_{n \rightarrow \infty} \left[ 1 - \left(\frac{3}{2}\right)^n \right] A_1
\end{aligned}$$

Assuming that  $\lim_{n \rightarrow \infty} \left(\frac{3}{4}\right)^n = 0$  and

$$\lim_{n \rightarrow \infty} \left(\frac{3}{2}\right)^n = 0$$

we will get the limit equal  $\langle 1 \rangle$ . We do not know how to explain the last limit but it seems that the solution passes through a solution to problem 1.

## 2 The arithmetic of solids in $\mathbb{R}^3$

Let us take any tetrahedron in  $\mathbb{R}^3$  and let us denote it by the symbol  $\langle 1 \rangle$ . Then the congruent tetrahedron with  $n$  times bigger sides (we denote it by the symbol  $\langle n \rangle$ ) we can present as follows

$$\langle n \rangle = \frac{n(n+1)(n+2)}{6} \langle 1 \rangle + \frac{(n-1)n(n+1)}{6} \langle D_1 \rangle + \frac{(n-2)(n-1)n}{6} \langle e_1 \rangle,$$

where  $\langle 1 \rangle, \langle D_1 \rangle, \langle e_1 \rangle$  are parts of the same parallelepiped cut by parallel planes passing through the respective vertices (Fig. 27, 28).

Let us mark by the symbol  $\langle e_n \rangle$  the tetrahedron congruent to the tetra-



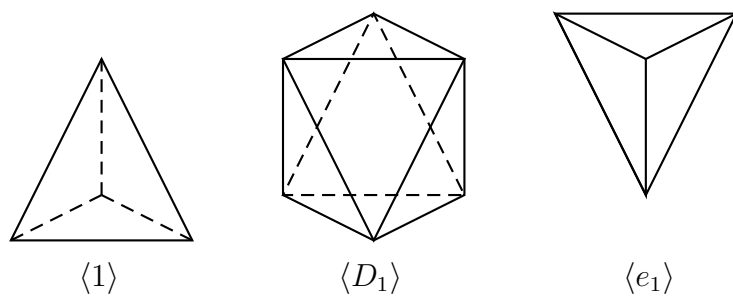


Fig. 27.

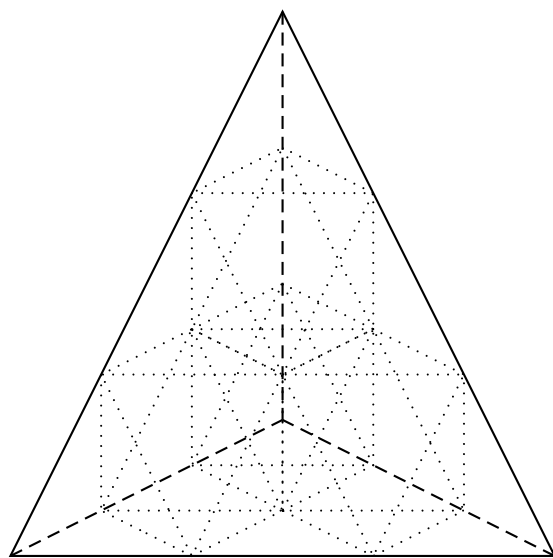


Fig. 28. Tetrahedron  $\langle 3 \rangle$ .

hedron  $\langle e_1 \rangle$  with  $n$  times bigger sides. Then

$$\begin{aligned}
\langle e_n \rangle &= \frac{n(n-1)(n-2)}{6} \langle 1 \rangle + \frac{(n-1)n(n+1)}{6} \langle D_1 \rangle + \frac{n(n+1)(n+2)}{6} \langle e_1 \rangle \\
&= -\frac{-n(-n+1)(-n+2)}{6} \langle 1 \rangle - \frac{(-n+1)(-n)(-n-1)}{6} \langle D_1 \rangle \\
&\quad - \frac{(-n)(-n-1)(-n-2)}{6} \langle e_1 \rangle \\
&= -\langle -n \rangle
\end{aligned}$$

We will denote each point of the plane  $\mathbb{R}^3$  by  $\langle 0 \rangle$ . The symbol  $-\langle n \rangle$ , where  $n \in \mathbb{Z}$  denotes the tetrahedron, which coinciding with the tetrahedron  $\langle n \rangle$  gives the empty set. It is easy to proof that below equation

$$\forall n \in \mathbb{N} \quad \langle n \rangle \cdot \langle m \rangle = \langle m \rangle \cdot \langle n \rangle = \langle nm \rangle$$

holds if the following conditions are fulfilled:

$$\begin{aligned}
\langle 1 \rangle \langle e_1 \rangle &= \langle e_1 \rangle, & \langle 1 \rangle \langle D_1 \rangle &= \langle D_1 \rangle, \\
\langle e_1 \rangle \langle e_1 \rangle &= \langle 1 \rangle, & \langle e_1 \rangle \langle D_1 \rangle &= \langle D_1 \rangle, \\
\langle D_1 \rangle \langle D_1 \rangle &= 4\langle 1 \rangle + 4\langle e_1 \rangle + 2\langle D_1 \rangle.
\end{aligned}$$

Let us transform

$$\begin{aligned}
\langle n \rangle &= n^3 \frac{\langle 1 \rangle + \langle D_1 \rangle + \langle e_1 \rangle}{6} + n^2 \frac{\langle 1 \rangle - \langle e_1 \rangle}{2} \\
&\quad + n \frac{2\langle 1 \rangle - \langle D_1 \rangle + 2\langle e_1 \rangle}{6} \\
&= n^3 A_3 + n^2 A_2 + n A_1.
\end{aligned} \tag{19}$$

The elements  $A_3 = \frac{\langle 1 \rangle + \langle D_1 \rangle + \langle e_1 \rangle}{6}$ ,  $A_2 = \frac{\langle 1 \rangle - \langle e_1 \rangle}{2}$ ,  $A_1 = \frac{2\langle 1 \rangle - \langle D_1 \rangle + 2\langle e_1 \rangle}{6}$  are orthogonal so the form (19) is orthogonal form of the element  $\langle n \rangle$ . The arithmetic form is following

$$\langle n \rangle = \frac{n(n-1)(n-2)}{6} \langle 3 \rangle + 3 \frac{n(n-1)(n-3)}{6} \langle 2 \rangle + 3 \frac{n(n-2)(n-3)}{6} \langle 1 \rangle.$$

The set  $\mathbb{N}_3 = \{\pm \langle n \rangle; n \in \mathbb{Z}\}$  is the subset of the ring

$$\begin{aligned}
\mathbb{P}_3(\mathbb{Q}) &= \{(x, y, z) = x\langle 1 \rangle + y\langle D_1 \rangle + z\langle e_1 \rangle = \alpha\langle 1 \rangle + \beta e + \gamma f + \delta g, \\
&\quad x, y, z, \alpha, \beta, \gamma, \delta \in \mathbb{Q}\}
\end{aligned}$$

where  $e = \langle e_1 \rangle = A_3 - A_2 + A_1$ ,  $f = -A_3 + A_2 + A_1$ ,  $g = A_3 + A_2 - A_1$ .  
Let us put the addition closed in  $\mathbb{N}_3$

$$\begin{aligned} \forall n, k, l, m \in \mathbb{Z} \\ \langle n + k + l + m \rangle &= \langle n + k + l \rangle + \langle n + k + m \rangle + \langle n + l + m \rangle \\ &\quad + \langle k + l + m \rangle - \langle n + k \rangle - \langle n + l \rangle \\ &\quad - \langle n + m \rangle - \langle k + l \rangle - \langle k + m \rangle - \langle l + m \rangle \\ &\quad + \langle n \rangle + \langle k \rangle + \langle l \rangle + \langle m \rangle, \end{aligned} \quad (20)$$

The operation is well-defined because it is easy to check that

$$\begin{aligned} \forall n, k, l, m \in \mathbb{Z} \quad \forall i = 1, 2, 3 \\ (n + k + l + m)^i &= (n + k + l)^i + (n + k + m)^i + (n + l + m)^i \\ &\quad + (k + l + m)^i - (n + k)^i - (n + l)^i \\ &\quad - (n + m)^i - (k + l)^i - (k + m)^i - (l + m)^i \\ &\quad + (n)^i + (k)^i + (l)^i + (m)^i \end{aligned}$$

As in  $\mathbb{N}_2$  as well as in  $\mathbb{N}_3$  we can reduce the number of components in the addition (20). If we put  $l = n + k$ ,  $m = 2n + k$  in the equation (20) we will get

$$\begin{aligned} \langle 4n + 3k \rangle &= \langle 4n + 2k \rangle + \langle 3n + 3k \rangle - \langle 3n + k \rangle \\ &\quad - \langle n + 2k \rangle + \langle n \rangle + \langle k \rangle \end{aligned}$$

For  $n = k = 1$  we will receive

$$\langle 7 \rangle = 2\langle 6 \rangle - \langle 4 \rangle - \langle 3 \rangle + 2\langle 1 \rangle. \quad (21)$$

The equation (22) is true only in an arithmetic sense. With the tetrahedrons  $\langle 6 \rangle$ ,  $\langle 6 \rangle$ ,  $-\langle 4 \rangle$ ,  $-\langle 3 \rangle$ ,  $\langle 1 \rangle$ ,  $\langle 1 \rangle$  we do not build (for example, virtually) the tetrahedron  $\langle 7 \rangle$ .

### 3 The arithmetic of solids in $\mathbb{R}^n$

In this section we will extend the concept of numbers  $\langle n \rangle$  for higher dimensions. To this effect we should remind some ideas.

**Definition 1.** *The Eulerian numbers denoted by symbol  $\langle m \rangle_k$  or  $A_{m,k}$  for  $k = 0, 1, 2, \dots, m-1$  give the number of permutation  $\pi_1 \pi_2 \dots \pi_m$  of the set  $\{1, 2, \dots, m\}$  having  $k$  ascents, that means  $k$  positions with  $\pi_j < \pi_{j+1}$ .*

We can get the Eulerian numbers:

a) by recurrence relation

$$\langle 1 \rangle_0 = 1 \quad \langle m \rangle_k = (m-k) \langle m-1 \rangle_{k-1} + (k+1) \langle m-1 \rangle_k$$

with assumption  $\langle m \rangle_k = 0$  for  $k < 0$ ,

b) explicit by the sum

$$\langle m \rangle_k = \sum_{i=0}^k \binom{m+1}{i} (k+1-i)^m (-1)^i.$$

The falling factorial  $x^{\underline{m}}$  is defined [5] by

$$x^{\underline{m}} = x(x-1)(x-2) \dots (x-m+1), \quad x^{\underline{0}} = 1 \quad \text{for natural } m \geq 0.$$

Let us remind Worpitzky's identity, which is true for  $m, n \in \mathbb{N}$ :

$$n^m = \sum_{k=0}^{m-1} \langle m \rangle_k \binom{n+k}{m} \quad (22)$$

Because

$$\langle m \rangle_m = 0, \quad \langle m \rangle_k = \langle m \rangle_{m-1-k}$$

and

$$\binom{a}{m} = \frac{(a-m+1)(a-m+2) \dots a}{m!} = \frac{a^{\underline{m}}}{m!}$$

then the pattern (22) will changes itself

$$\begin{aligned} n^m &= \sum_{k=0}^{m-1} \left\langle \begin{matrix} m \\ k \end{matrix} \right\rangle \binom{n+k}{m} \\ &= \sum_{k=0}^{m-1} \left\langle \begin{matrix} m-1-k \\ k \end{matrix} \right\rangle \binom{n+k}{m} \end{aligned}$$

at this point we count from the end

$$\begin{aligned} \text{inserting } m-k-1 \text{ in place of } k &= \sum_{k=0}^{m-1} \left\langle \begin{matrix} m \\ k \end{matrix} \right\rangle \binom{n+m-k-1}{m} \\ &= \sum_{k=1}^m \left\langle \begin{matrix} m \\ k-1 \end{matrix} \right\rangle \binom{n+m-k}{m} \\ &= \sum_{k=1}^m \left\langle \begin{matrix} m \\ k-1 \end{matrix} \right\rangle \frac{(n+m-k)^{\underline{m}}}{m!} \quad (23) \end{aligned}$$

According to [6, 7] the Eulerian number fulfills:

$$\left\langle \begin{matrix} m \\ k \end{matrix} \right\rangle = m! V(m, k),$$

where  $V(m, k)$  is the volume of the  $k$ -th slice of the cube  $[0, 1]^m$  located between two successive parallel planes with normal vector  $[1, 1, \dots, 1]$ , distant from the origin of values  $\frac{k}{\sqrt{m}}$  and  $\frac{k+1}{\sqrt{m}}$ .

For example, for  $m = 2$  the cube  $[0, 1]^2$  has two slices-triangles (Fig. 29) with

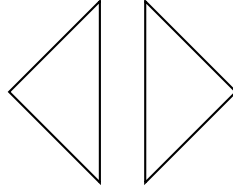


Fig. 29.

the volumes  $V(2, 0) = \frac{1}{2}$ ,  $V(2, 1) = \frac{1}{2}$ .

For  $m = 3$  there are three slices (Fig. 27) with the volumes

$V(3, 0) = \frac{1}{6}$ ,  $V(3, 1) = \frac{4}{6}$ ,  $V(3, 2) = \frac{1}{6}$ .

Let us replace in (23) each Eulerian number  $\langle \begin{smallmatrix} m \\ k-1 \end{smallmatrix} \rangle = m!V(m, k-1)$  by the expresion  $\langle 1_k \rangle$ . Then we can write

$$\langle n_1 \rangle = \sum_{k=1}^m \frac{(n+m-k)^m}{m!} \langle 1_k \rangle$$

where  $\langle n_1 \rangle$  is a symbol of m-dimensional simplex with sides  $n$  times bigger than the sides of the simplex  $\langle 1_1 \rangle$ . Of course a volume of the simplex  $\langle n_1 \rangle$  is equal to  $n^m$ .

Let us transform  $\langle n_1 \rangle$  to polynomial form grouping by powers of  $n$ .

$$\langle n_1 \rangle = \sum_{k=1}^m n^{m-k+1} A_{m-k+1}$$

If we assume an orthogonality of elements  $A_{m-k+1}$  then the following condition will be satisfied

$$\forall n, m \in \mathbb{N} \quad \langle n_1 \rangle \cdot \langle m_1 \rangle = \langle (nm)_1 \rangle$$

We can also introduce an addition closed in the set  $N_n = \{\langle n_1 \rangle; n \in \mathbb{Z}\}$

$$\left\langle \sum_{i=1}^{m+1} a_i \right\rangle = \sum_{j=1}^{m+1} \left\langle \sum_{\substack{i=1 \\ i \neq j}}^{m+1} a_i \right\rangle - \sum_{\substack{j,h \\ j \neq h}} \left\langle \sum_{\substack{i=1 \\ i \neq j,h}}^{m+1} a_i \right\rangle + \cdots (-1)^{m+1} \sum_{i=1}^{m+1} \langle a_i \rangle$$

The general case of solids  $\langle n_1 \rangle$  is only touched here and requires further study.

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